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The evolution of a weakly nonlinear, weakly damped, capillary–gravity wave packet

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Longuet-Higgins's (1976) analysis of energy transfer within a narrow spectrum of gravity waves with approximately uncorrelated phases is generalized to accommodate capillarity and weak damping. The analysis is based on the corresponding generalization of Zakharov's (1968) evolution equation for weakly nonlinear, deep-water gravity-wave packets. The results for a symmetric normal spectrum are expressed in terms of elliptic integrals and depend, after appropriate scaling, on a single similarity parameter and on the sign of the curvature of the linear dispersion relation. Energy transfer is away from the peak of that spectrum if $kl_* < 0.393$, where k is the wavenumber and l_* is the capillary length (2.8 mm for water), but may be towards the peak if $0.343 < kl_* < 0.707$ (4.5 cm $> 2\pi/k >$ 2.5 cm for water). The formulation is based on energy exchange through resonant quartets and is not valid in the neighbourhood of $kl_* = 0.707$, at which the second harmonic of a capillary–gravity wave resonates with its fundamental (Wilton's ripples). The modulational instability of a weakly damped capillary–gravity wave is examined in an Appendix.

1. Introduction

Longuet-Higgins (1976, hereinafter referred to as LH followed by the appropriate equation number or section) shows that the action density $(\omega_0/k_0^5)N(\boldsymbol{\kappa})$ at the wavenumber

$$\boldsymbol{k} = (k_0, 0) + \epsilon k_0 \boldsymbol{\kappa}, \quad \boldsymbol{\kappa} = (\lambda, \mu), \tag{1.1 a, b}$$

and frequency

$$\omega = \omega_0(1+\epsilon^2\sigma) \tag{1.1 c}$$

of weakly nonlinear, deep-water gravity waves with approximately uncorrelated phases in a reference frame moving in the primary (x) direction with the group velocity $\mathrm{d}\omega_0/\mathrm{d}k_0$ is governed by (LH, (4.11))

$$\frac{\mathrm{d}N_1}{\mathrm{d}\tau} = 4\pi \int \dots \int [(N_1+N_2)N_3N_4 - (N_3+N_4)N_1N_2] \times \delta(\sigma_1+\sigma_2-\sigma_3-\sigma_4)\,\delta(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2-\boldsymbol{\kappa}_3-\boldsymbol{\kappa}_4)\,\mathrm{d}\boldsymbol{\kappa}_2\,\mathrm{d}\boldsymbol{\kappa}_3\,\mathrm{d}\boldsymbol{\kappa}_4, \tag{1.2 a}$$

wherein

$$\tau = \epsilon^2\omega_0 t \tag{1.2 b}$$

is a slow time, δ is Dirac's delta function, and $\mathrm{d}\boldsymbol{\kappa}_n \equiv \mathrm{d}\lambda_n\,\mathrm{d}\mu_n$. The parameter ϵ is a measure of the wave slope, and the implicit limit in the derivation of (1.2) is $\epsilon \downarrow 0$ with $\boldsymbol{\kappa}$, σ and $\tau = O(1)$. Longuet-Higgins also shows that the flow of energy away from the peak of the symmetric normal spectrum (here $Q \equiv 2Q_{\mathrm{LH}}$)

$$N(\boldsymbol{\kappa}) = N_0\,\mathrm{e}^{-\frac{1}{2}(P\lambda^2+Q\mu^2)} \tag{1.3}$$

is governed by (LH, (9.3))

$$\frac{\mathrm{d}N}{\mathrm{d}\tau} = -\frac{N_0^3}{(\frac{1}{2}PQ)^{\frac{1}{2}}} H\left(\frac{2P}{Q}\right), \tag{1.4}$$

where $H(x) = H(1/x)$ is a function that decays slowly from 65.82 at $x = 1$ to 0 at $x = 0$ (LH, table 1, after correcting for a missing factor of 2). I present here the generalizations of (1.2) and (1.4) to weakly damped, capillary–gravity waves.

The changes in (1.2) and (1.4) due to capillarity without damping may be inferred from similarity considerations. Expanding $k \equiv |\boldsymbol{k}|$ and the linear dispersion relation $\omega = \omega_0(k)$ about $\epsilon = 0$ and neglecting $O(\epsilon^3)$, we obtain

$$k \approx k_0(1+\epsilon\lambda+\tfrac{1}{2}\epsilon^2\mu^2), \quad \omega \approx \omega_0+\epsilon\omega_0' k_0\lambda+\tfrac{1}{2}\epsilon^2(k_0^2\omega_0''\lambda^2+k_0\omega_0'\mu^2), \tag{1.5a, b}$$

wherein $\omega_0 \equiv \omega_0(k_0)$. The effect of observing the envelope (i.e. the complex amplitude of the carrier $\exp[\mathrm{i}(k_0 x-\omega_0 t)]$) in a reference frame moving in the x-direction with the group velocity ω_0' is to shift the frequency by $\omega_0'\epsilon k_0\lambda$ ($\epsilon k_0\lambda$ is the x-component of $\boldsymbol{k}-\boldsymbol{k}_0$), whence the dimensionless frequency of the envelope on the hypothesis of infinitesimal amplitude is (I use σ where Longuet-Higgins uses ω)

$$\sigma \equiv \frac{\omega-\omega_0'\epsilon k_0\lambda}{\epsilon^2\omega_0} = L\lambda^2+M\mu^2, \tag{1.6}$$

where
$$L \equiv \frac{k_0^2\omega_0''}{2\omega_0}, \quad M \equiv \frac{k_0\omega_0'}{2\omega_0}. \tag{1.7a, b}$$

Amplitude (nonlinear) dispersion for a monochromatic wave of complex amplitude a adds a term $Ck^2|a|^2\omega_0$ to ω (C is the Landau constant) and a corresponding term to σ; however, this term may be evaluated at $k = k_0$ and therefore is independent of λ and μ and does not affect $\sigma_1+\sigma_2-\sigma_3-\sigma_4$ in (1.2).

Nonlinearity also enters the amplitude-evolution equation, which would be

$$a_t = -\mathrm{i}C\omega_0 k_0^2|a|^2 a \tag{1.8}$$

for a monochromatic wave of slowly varying amplitude and is given by LH, (4.3) for a gravity-wave packet, for which $C = \frac{1}{2}$. Introducing the factor $2C$ on the right-hand side of LH, (4.3), we find that $4C^2$ appears on the right-hand side of LH, (4.11). It follows that capillarity may be accommodated in (1.2) by introducing $4C^2$ on the right-hand side thereof and calculating $\sigma_1+\sigma_2-\sigma_3-\sigma_4$ from (1.6) in place of LH, (3.2), in which L and M (in the present notation) assume their gravity-wave values $-\frac{1}{8}$ and $\frac{1}{4}$, respectively. It then follows from similarity considerations† that the counterpart of (1.4) for an undamped surface wave characterized by the linear dispersion relation $\omega = \omega_0(k)$ and the Landau constant $C(k)$ must be of the form

$$\frac{\partial N}{\partial \tau} = -\frac{C^2 N_0^3}{|LMPQ|^{\frac{1}{2}}} H\left(\left|\frac{MP}{LQ}\right|\right), \tag{1.9}$$

which reduces to (1.4) for $C = \frac{1}{2}$, $L = -\frac{1}{8}$ and $M = \frac{1}{4}$; however, H depends on $\operatorname{sgn} L$ (M is positive-definite – see §2), and Longuet-Higgins's result for H may be used in (1.9) only for $L < 0$. I obtain the corresponding result for $L > 0$ ($2\pi/k < 4.5$ cm for water) in §4 and show that energy transfer is towards the peak in part of this domain.

† These similarity arguments also may be used to generalize Fox's (1976) results.

The effects of weak damping require a more detailed analysis for their derivation. A heuristic procedure, suggested by the solution of the Landau equation (cf. (1.8))

$$a_t + \omega_0 Da = -\mathrm{i}C\omega_0 k^2 |a|^2 a, \tag{1.10}$$

where D is the damping ratio (of actual to critical damping), is to replace the operator $\mathrm{d}/\mathrm{d}\tau$ in (1.2) by $(\mathrm{d}/\mathrm{d}\tau)+2\alpha$, wherein

$$\alpha \equiv \frac{D}{\epsilon^2} \tag{1.11}$$

is scaled to complement the scaling of $\tau = \epsilon^2\omega_0 t$, and the factor of 2 reflects the fact that N is proportional to the square of the amplitude. The change of variable

$$N = \mathrm{e}^{-2\alpha\tau}\hat{N}(T), \quad T = \frac{1-\mathrm{e}^{-4\alpha\tau}}{4\alpha}, \tag{1.12a, b}$$

together with the incorporation of the factor $4C^2$ (see above), then yields

$$\frac{\mathrm{d}\hat{N}_1}{\mathrm{d}T} = 16\pi C^2 \int\cdots\int [(\hat{N}_1+\hat{N}_2)\hat{N}_3\hat{N}_4 - (\hat{N}_3+\hat{N}_4)\hat{N}_1\hat{N}_2] \times \delta(\sigma_1+\sigma_2-\sigma_3-\sigma_4)\,\delta(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2-\boldsymbol{\kappa}_3-\boldsymbol{\kappa}_4)\,\mathrm{d}\boldsymbol{\kappa}_2\,\mathrm{d}\boldsymbol{\kappa}_3\,\mathrm{d}\boldsymbol{\kappa}_4, \tag{1.13}$$

which proves valid for $\alpha \downarrow 0$ with $\alpha\tau$ fixed and implies $N = O(\alpha^{-1}\,\mathrm{e}^{-2\alpha\tau})$ as $\alpha\tau \uparrow \infty$.

I proceed as follows. In §2, I derive the evolution equation for the envelope of a weakly nonlinear, weakly damped, deep-water, capillary–gravity wave packet. The end result, (2.10), is a generalization (to incorporate damping) of Djordjevic & Redekopp's (1977) generalization (to incorporate capillarity) of Zakharov's (1968) and Davey & Stewartson's (1974) equations. The spatial operator in these evolution equations is elliptic/hyperbolic for $L \gtrless 0$.

In §3, I posit the envelope as a Fourier superposition of elementary waves with amplitudes that are densely distributed over the $\boldsymbol{\kappa}$-spectrum and phases that are *approximately* uncorrelated. (This last approximation is, in effect, a closure hypothesis.) The analysis follows LH, §4, with $\mathrm{d}/\mathrm{d}\tau$ replaced by $\mathrm{d}/\mathrm{d}\tau+2\alpha$ when operating on action and $\delta(\sigma_1+\sigma_2-\sigma_3-\sigma_4)$ replaced by a response function that is asymptotic (as $\alpha\tau \uparrow \infty$) to a simple resonance curve (the Witch of Angesi) with a half-power bandwidth of 4α. I then let $\alpha \downarrow 0$ with T fixed to obtain (1.13).

In §4, I invoke (1.3) and express the counterpart of H in (1.19) for both $L > 0$ and $L < 0$ in terms of elliptic integrals.

The evolution equation (2.10) also may be used to determine the effect of weak ($\alpha = O(1)$) damping on modulational (sideband or Benjamin–Feir) instability. The results appear to be worth recording but are of only peripheral interest in the present context, and I therefore have relegated their derivation to Appendix C.

The present results are limited by the narrow-band approximation ($\epsilon \downarrow 0$ in (1.1)). Dungey & Hui (1979) have obtained the second approximation for a gravity-wave spectrum of small but finite bandwidth and have shown that finite-bandwidth effects are essential in the analysis of the JONSWAP spectra (Hasselmann *et al.* 1973). It would be desirable to extend Dungey & Hui's analysis to include capillarity and damping, but the anticipated algebraic complexity of the coupling coefficient $G(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \boldsymbol{\kappa}_3, \boldsymbol{\kappa}_4)$ (which reduces to a constant in the limit $\epsilon \downarrow 0$) is forbidding.

2. The envelope-evolution equation

The dispersion relation for a weakly nonlinear, weakly damped, deep-water capillary–gravity wave of the form

$$\zeta \equiv \mathrm{Re} Z = \mathrm{Re}\{a\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}\}, \tag{2.1}$$

where ζ is the free-surface displacement, Z is its complex counterpart, Re implies *the real part of*, and a is a complex amplitude, is given by†

$$\omega = \omega_0(k)[1+Ck^2|Z|^2-\mathrm{i}D] \equiv \Omega(k, k|Z|), \tag{2.2}$$

where
$$\omega_0(k) = (gk+Tk^3)^{\frac{1}{2}}, \quad k \equiv |\boldsymbol{k}|, \tag{2.3a,b}$$

$$C = \frac{8+\kappa^2+2\kappa^4}{16(1+\kappa^2)(1-2\kappa^2)}, \quad \kappa = k\left(\frac{T}{g}\right)^{\frac{1}{2}} \equiv kl_*, \tag{2.4a,b}$$

$$D = \frac{2\nu k^2}{\omega_0} \equiv (kl_\nu)^2 \quad (kl_\nu \ll 1), \tag{2.5a}$$

T is the surface tension divided by the density, C is the Landau constant, $l_* = (T/g)^{\frac{1}{2}}$ is the capillary length, ν is the kinematic viscosity and $l_\nu = (2\nu/\omega_0)^{\frac{1}{2}}$ is a viscous length. By 'weakly nonlinear' and 'weakly damped', we imply $k^2|a|^2 \ll 1$ and $D = O(k^2|a|^2)$.

The damping ratio (2.5a) is based on the assumptions of a clean surface and a thin boundary layer (Lamb 1932, §348). The damping ratio for a contaminated surface lies between $(kl_\nu)^2$ and $\frac{1}{2}kl_\nu$ (Miles 1967) and is given by (Lamb 1932, §351)

$$D = \tfrac{1}{4}kl_\nu \tag{2.5b}$$

for a saturated (inextensible) surface. Rewriting (2.5a,b) as

$$D = 2\nu g^{\frac{1}{4}}T^{-\frac{3}{4}}\kappa^{\frac{3}{2}}(1+\kappa^2)^{-\frac{1}{2}} \tag{2.6a}$$

and
$$D = 2^{-\frac{3}{2}}\nu^{\frac{1}{2}}g^{\frac{1}{8}}T^{-\frac{3}{8}}\kappa^{\frac{3}{4}}(1+\kappa^2)^{-\frac{1}{4}}, \tag{2.6b}$$

we obtain $D = 0.0021$ and 0.011 respectively, for water ($\nu = 0.010$, $g = 980$, $T = 79$ in c.g.s. units) at $\kappa^2 = \frac{1}{2}$.

The evolution equation for the complex wave packet

$$Z = \epsilon k_0^{-1} \mathscr{A}\, \mathrm{e}^{\mathrm{i}(k_0 x-\omega_0 t)}, \tag{2.7}$$

where $\mathscr{A}$ is a dimensionless, slowly varying complex envelope and $\omega_0 \equiv \omega_0(k_0)$, may be posed in the form

$$Z_t = -\mathrm{i}\Omega(K, \epsilon|\mathscr{A}|)Z, \quad \mathrm{i}K \equiv (\partial_x^2+\partial_y^2)^{\frac{1}{2}}, \tag{2.8a,b}$$

which is a generalization of $Z_t = -\mathrm{i}\omega Z$ for the elementary wave (2.1); cf. Davey (1972) and Whitham (1974, §17.7). Introducing the slow variables

$$\xi = \epsilon k_0(x-\omega_0' t), \quad \eta = \epsilon k_0 y, \quad \tau = \epsilon^2\omega_0 t \tag{2.9a,b,c}$$

in a reference frame moving with the group velocity $\omega_0' \equiv \omega_0'(k_0)$, expanding the operators $\mathrm{i}\Omega$ and $\mathrm{i}K$ about $\mathrm{i}\omega_0$ and $\mathrm{i}k_0$, respectively, and letting $\epsilon \downarrow 0$ with $\mathscr{A}, \xi, \eta, \tau = O(1)$ and $D = O(\epsilon^2)$, we obtain

$$[\mathrm{i}(\partial_\tau+\alpha)+L\,\partial_\xi^2+M\,\partial_\eta^2-C|\mathscr{A}|^2]\mathscr{A} = 0, \tag{2.10}$$

† I recollect that this result is due to Harrison (1909), but I have been unable (at this time) to obtain the original reference.

where $\alpha \equiv D/\epsilon^2$ (1.11) and L and M, defined as in (1.7), are given by

$$L = \frac{k_0^2\,\omega_0''}{2\omega_0} = -\frac{(1-6\ell^2-3\ell^4)}{8(1+k^2)^2}, \quad M = \frac{k_0\,\omega_0'}{2\omega_0} = \frac{1}{4}\left(\frac{1+3\ell^2}{1+\ell^2}\right). \tag{2.11a, b}$$

Setting $\alpha = 0$ and allowing for differences in notation, we recover Djordjevic & Redekopp's (1977) evolution equation.

The parameters $|L|$, M and $|C|$, but not sgn L and sgn C, may be eliminated from (2.10) by rescaling ξ, η and $\mathscr{A}$, respectively. We note that $C \gtrless 0$ for $\ell \lessgtr 0.707$ and $L \gtrless 0$ for $\ell \gtrless 0.393$ and that C, L and M vary through $(\frac{1}{2}, \pm\infty, -\frac{1}{16})$, $(-\frac{1}{8}, 0, \frac{3}{8})$ and $(\frac{1}{4}, \frac{3}{4})$, respectively, as ℓ varies from 0 to ∞. The dispersion relation (2.2) and the evolution equation (2.10) obviously break down in the neighbourhood of $\ell^2 = \frac{1}{2}$, for which the elementary wave (2.1) resonates with its second harmonic (Wilton's ripples). Moreover, without damping, self-focusing may occur for $\ell^2 > \frac{1}{2}$ (Ablowitz & Segur 1979), although it seems likely that damping vitiates this singular behaviour. In any event, the domain of principal interest for waves in the open sea is $\ell^2 < \frac{1}{2}$.

3. The spectrum-evolution equation

We pose the general solution of (2.10) for an unbounded domain in the form

$$\mathscr{A}(\xi,\eta,\tau) = a_n(\tau)\exp\left\{\mathrm{i}\left[\lambda_n\xi+\mu_n\eta-\int_0^\tau \sigma_n(\tau)\,\mathrm{d}\tau\right]\right\}, \tag{3.1}$$

where the summation convention holds for repeated indices except as noted, the summation is over the $\boldsymbol{\kappa} \equiv (\lambda_n, \mu_n)$ spectrum,

$$\sigma_n = L\lambda_n^2 + M\mu_n^2 + \beta(\tau), \tag{3.2}$$

and β, which represents amplitude dispersion, is real and independent of n (see (A 7) below). Substituting (3.1) into (2.10), invoking (3.2), and equating coefficients of $\exp[\mathrm{i}(\lambda_n\xi+\mu_n\eta)]$, we obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau}+\alpha\right)a_n = \mathrm{i}[\beta a_n - C\delta_{jl\overline{mn}}\,a_j\,a_l\,a_m^*\,e_{jl\overline{mn}}], \tag{3.3$_n$}$$

where

$$\delta_{jl\overline{mn}} = \begin{matrix}1\\0\end{matrix} \quad \text{for } \boldsymbol{\kappa}_j+\boldsymbol{\kappa}_l-\boldsymbol{\kappa}_m-\boldsymbol{\kappa}_m \begin{matrix}=\\ \neq\end{matrix} 0, \tag{3.4}$$

$$e_{jl\overline{mn}} \equiv \exp(-\mathrm{i}\sigma_{jl\overline{mn}}\tau), \quad \sigma_{jl\overline{mn}} \equiv \sigma_j+\sigma_l-\sigma_m-\sigma_n, \tag{3.5a, b}$$

a_n^* is the complex conjugate of a_n, and, here and subsequently, the subscript n on an equation number implies that n is not summed in that equation. The condition $\boldsymbol{\kappa}_j+\boldsymbol{\kappa}_l = \boldsymbol{\kappa}_m+\boldsymbol{\kappa}_n$ implies that $\boldsymbol{\kappa}_j, \dots \boldsymbol{\kappa}_n$ are the corners of a parallelogram in the $\boldsymbol{\kappa}$-plane. The quartet is resonant for $\sigma_j+\sigma_l = \sigma_m+\sigma_n$, and $\boldsymbol{\kappa}_j, \dots \boldsymbol{\kappa}_n$ then lie on an ellipse/hyperbola for $L \gtrless 0$.

We now suppose that the a_n in (3.1) are densely distributed in the $\boldsymbol{\kappa} \equiv (\lambda, \mu)$-plane with approximately uncorrelated phases, such that (cf. LH, (4.2))

$$A_n \equiv \tfrac{1}{2}\overline{|a_n|^2} \sim N(\boldsymbol{\kappa}, \tau), \tag{3.6}$$

where the overbar signifies an ensemble average, and N is a dimensionless action density, for which the dependence on τ is henceforth implicit. The total action is given by (Longuet-Higgins uses A for his envelope and M for the total action)

$$A \equiv \tfrac{1}{2}\overline{a_n a_n^*} \sim \iint N(\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa} \quad (\mathrm{d}\boldsymbol{\kappa} \equiv \mathrm{d}\lambda\,\mathrm{d}\mu). \tag{3.7}$$

Multiplying (3.3) by a_n^*, adding the complex conjugate of the result, and averaging, we obtain (cf. LH, (4.4), in which $C_n \equiv 2A_n$ in the present notation)

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau}+2\alpha\right)A_n = -C\delta_{jl\overline{mn}}\,\mathrm{Re}\,\{\mathrm{i}B_{jl\overline{mn}}\,e_{jl\overline{mn}}\}, \qquad (3.8)_n$$

where
$$B_{jl\overline{mn}} \equiv \overline{a_j\,a_l\,a_m^*\,a_n^*}. \tag{3.9}$$

Summing (3.8) over n, invoking (3.7), and remarking that the quadruple sum $B_{jl\overline{mn}}\,e_{jl\overline{mn}}$ is real (which may be confirmed by interchanging j and m and l and n and invoking $e_{mn\overline{jl}} = e^*_{jl\overline{mn}}$), we obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau}+2\alpha\right)A = 0. \tag{3.10}$$

It follows that the total action decays according to

$$A = \hat{A}\,\mathrm{e}^{-2\alpha\tau}, \tag{3.11}$$

where $\hat{A}$ is a constant. Corresponding results for momentum and energy are derived in Appendix A.

Proceeding as in the derivation of LH, (4.7), we construct $(\mathrm{d}/\mathrm{d}\tau+4\alpha)\,B_{jl\overline{mn}}$ with the aid of (3.3) and invoke the approximation of uncorrelated phases to simplify the result (cf. LH, (4.5) and the immediately following argument), which then may be reduced to

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau}+4\alpha\right)B_{jl\overline{mn}} = -16\mathrm{i}C\delta_{jl\overline{mn}}\,\Gamma_{jlmn}\,e_{mn\overline{jl}}, \qquad (3.12)_{jlmn}$$

where
$$\Gamma_{jlmn} = (A_j+A_l)\,A_m\,A_n-(A_m+A_n)\,A_j\,A_l. \tag{3.13}$$

The approximation of uncorrelated phases implies $B_{jl\overline{mn}} = 0$ unless either $j = m$ and $l = n$ or $j = n$ and $l = m$, whence, after invoking (3.6),

$$B_{jl\overline{mn}} = 4(\delta_{jm}\,\delta_{ln}+\delta_{jn}\,\delta_{lm})\,A_m\,A_n \qquad (3.14)_{jlmn}$$

in first approximation. The right-hand side of (3.8) then vanishes, and the corresponding approximations to A_n and Γ_{jlmn} are

$$A_n(\tau) = \hat{A}_n\,\mathrm{e}^{-2\alpha\tau}, \quad \Gamma_{jlmn}(\tau) = \hat{\Gamma}_{jlmn}\,\mathrm{e}^{-6\alpha\tau}, \qquad (3.15a,b)$$

where $\hat{A}_n$ and $\hat{\Gamma}_{jlmn}$ are constants. Substituting (3.15b) into (3.12), integrating, and choosing $\tau = 0$ (which corresponds to $\tau = -\infty$ in Longuet-Higgins's formulation) as a time at which the phases of the a_n are uncorrelated and (3.14) holds, we obtain the second approximation

$$B_{jl\overline{mn}}(\tau) = 4(\delta_{jm}\,\delta_{ln}+\delta_{jn}\,\delta_{lm})\,\hat{A}_m\,\hat{A}_n\,\mathrm{e}^{-4\alpha\tau}-16\mathrm{i}C\delta_{jl\overline{mn}}\,\hat{\Gamma}_{jlmn}\,\mathrm{e}^{-4\alpha\tau}\left[\frac{1-\mathrm{e}^{-(2\alpha-\mathrm{i}\sigma)\tau}}{2\alpha-\mathrm{i}\sigma}\right], \qquad (3.16)_{jlmn}$$

wherein $\sigma \equiv \sigma_{jl\overline{mn}}$. Substituting (3.16) into (3.8) and solving for A_n, we obtain the second approximation

$$A_n(\tau) = \mathrm{e}^{-2\alpha\tau}[\hat{A}_n-8C^2\delta_{jl\overline{mn}}\,\hat{\Gamma}_{jlmn}\,R(\sigma_{jl\overline{mn}},\tau)], \qquad (3.17)_n$$

where
$$R(\sigma,\tau)=\frac{1-2\,\mathrm{e}^{-2\alpha\tau}\cos\sigma\tau+\mathrm{e}^{-4\alpha\tau}}{4\alpha^2+\sigma^2} \tag{3.18}$$

is a response function for the $\boldsymbol{\kappa}_{j,l,m,n}$ quartet that grows like $\sin^2\sigma\tau$ for $\alpha\tau\ll 1$ and is asymptotic to a resonance curve with a half-power bandwidth of 4α for $\alpha\tau\gg 1$.

Invoking (3.6) and transforming the summations over j,l,m in (3.17) to integrals over $\boldsymbol{\kappa}_{2,3,4}$ as in (3.7), we obtain

$$N_1=\mathrm{e}^{-2\alpha\tau}\left[\hat{N}_1+8C^2\int\ldots\int\hat{\Gamma}_{1234}R(\sigma_1+\sigma_2-\sigma_3-\sigma_4,\tau)\,\delta(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2-\boldsymbol{\kappa}_3-\boldsymbol{\kappa}_4)\,\mathrm{d}\boldsymbol{\kappa}_2\,\mathrm{d}\boldsymbol{\kappa}_3\,\mathrm{d}\boldsymbol{\kappa}_4\right], \tag{3.19}$$

where
$$\hat{\Gamma}_{1234}=(\hat{N}_1+\hat{N}_2)\hat{N}_3\hat{N}_4-(\hat{N}_3+\hat{N}_4)\hat{N}_1\hat{N}_2,\quad \hat{N}_i\equiv\hat{N}(\boldsymbol{\kappa}_i). \tag{3.20a, b}$$

Following LH, §B, we let $\boldsymbol{\kappa}_{1,2}=\bar{\boldsymbol{\kappa}}\mp\boldsymbol{\kappa}'$, $\boldsymbol{\kappa}_{3,4}=\bar{\boldsymbol{\kappa}}\mp\boldsymbol{\kappa}''$, where $\bar{\boldsymbol{\kappa}}\equiv\frac{1}{2}(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2)=\frac{1}{2}(\boldsymbol{\kappa}_3+\boldsymbol{\kappa}_4)$ is the centroid of the parallelogram defined by $\boldsymbol{\kappa}_{1,2,3,4}$, or, equivalently,

$$\boldsymbol{\kappa}_2=\boldsymbol{\kappa}_1+2\boldsymbol{\kappa}',\quad \boldsymbol{\kappa}_3=\boldsymbol{\kappa}_1+\boldsymbol{\kappa}'-\boldsymbol{\kappa}'',\quad \boldsymbol{\kappa}_4=\boldsymbol{\kappa}_1+\boldsymbol{\kappa}'+\boldsymbol{\kappa}'', \tag{3.21a, b, c}$$

and carry out the $\boldsymbol{\kappa}_2$ integration, and invoke

$$J\begin{pmatrix}\boldsymbol{\kappa}_3 & \boldsymbol{\kappa}_4\\ \boldsymbol{\kappa}' & \boldsymbol{\kappa}''\end{pmatrix}=J\begin{pmatrix}\lambda_3 & \lambda_4\\ \lambda' & \lambda''\end{pmatrix}J\begin{pmatrix}\mu_3 & \mu_4\\ \mu' & \mu''\end{pmatrix}=4 \tag{3.22}$$

(LH, (B 1) gives 2 for this Jacobian, an error also noted by Dungey & Hui 1979) to reduce (3.19) to

$$N_1=\mathrm{e}^{-2\alpha\tau}\left[\hat{N}_1+32C^2\iiiint\hat{\Gamma}_{1234}R(\sigma,\tau)\,\mathrm{d}\boldsymbol{\kappa}'\,\mathrm{d}\boldsymbol{\kappa}''\right], \tag{3.23}$$

wherein
$$\sigma=\sigma_1+\sigma_3-\sigma_2-\sigma_4=2L(\lambda'^2-\lambda''^2)+2M(\mu'^2-\mu''^2). \tag{3.24}$$

We now suppose that $\alpha\ll 1$, so that the resonance is sharp. The integral in (3.23) then is dominated by the contributions from the neighbourhood of $\sigma=0$, and we may approximate $R(\sigma,\tau)$ by

$$R(\sigma,\tau)\sim\delta(\sigma)\int_{-\infty}^{\infty}R(\sigma,\tau)\,\mathrm{d}\sigma=2\pi\delta(\sigma)T(\tau)\quad(\alpha\downarrow 0),\dagger \tag{3.25}$$

where
$$T=\frac{1-\mathrm{e}^{-4\alpha\tau}}{4\alpha}, \tag{3.26}$$

as in (1.12*b*). Substituting (3.25) into (3.23), we obtain

$$N_1=\mathrm{e}^{-2\alpha\tau}\left[\hat{N}_1+64\pi C^2T\iiiint\hat{\Gamma}_{1234}\,\delta(\sigma)\,\mathrm{d}\boldsymbol{\kappa}'\,\mathrm{d}\boldsymbol{\kappa}''\right]\quad(\alpha\ll 1), \tag{3.27}$$

which is the integral (with respect to T) of (1.13) after invoking (3.21) and carrying out the integration with respect to $\boldsymbol{\kappa}_2$ therein.

† The implicit limit in (3.25) is $\alpha\downarrow 0$ with $\alpha\tau$ fixed. The limit $\alpha\downarrow 0$ with τ fixed yields $T\uparrow\tau$, but this approximation clearly is not uniformly valid as $\tau\uparrow\infty$.

pqr	134	234	123	124
C_1	4	8	6	6
C_2	0	0	−2	2
C_{11}	2	6	5	5
C_{12}	0	0	−2	2
C_{22}	2	2	1	1
R	4	8	$4\surd 2$	$4\surd 2$
S	2	$2\surd 3$	2	2

TABLE 1. The coefficients in (4.4) and (4.15)

4. Evolution of peak density

We posit the symmetric, normal spectrum

$$\hat{N}(\boldsymbol{\kappa}) = N_0 \exp\left(-\tfrac{1}{2}P\lambda^2 - \tfrac{1}{2}M\mu^2\right) \tag{4.1}$$

(cf. LH, (9.1), but here $Q \equiv 2Q_{\mathrm{LH}}$). Substituting (4.1) into (3.27) and letting $\boldsymbol{\kappa}_1 \equiv \boldsymbol{\kappa}$ and $\hat{N}_1 \equiv \hat{N}$, we obtain

$$N(\boldsymbol{\kappa}, \tau) = \mathrm{e}^{-2\alpha\tau}[\hat{N}(\boldsymbol{\kappa}) + G(\boldsymbol{\kappa})\hat{N}^3(\boldsymbol{\kappa})T(\tau)], \tag{4.2}$$

where
$$G = G_{134} + G_{234} - G_{123} - G_{124}, \tag{4.3}$$

$$G_{pqr} = 64\pi C^2 \iiiint \exp\left[-\tfrac{1}{2}PF_{pqr}(\lambda, \lambda', \lambda'') - \tfrac{1}{2}QF_{pqr}(\mu, \mu', \mu'')\right]\delta(\sigma)\,\mathrm{d}\boldsymbol{\kappa}'\,\mathrm{d}\boldsymbol{\kappa}'', \tag{4.4}$$

σ is given by (3.24), and

$$\begin{aligned} F_{pqr}(\lambda, \lambda', \lambda'') &= \lambda_p^2 + \lambda_q^2 + \lambda_r^2 - 3\lambda^2 &\quad (4.5a)\\ &= C_1\lambda\lambda' + C_2\lambda\lambda'' + C_{11}\lambda'^2 + C_{12}\lambda'\lambda'' + C_{22}\lambda''^2. &\quad (4.5b)\end{aligned}$$

The coefficients $C_1, \ldots, C_{22}$ are given in table 1.

The reduction of the integral in (4.4) depends on $\operatorname{sgn} L$. Assuming $L > 0$, we introduce the elliptic-polar coordinates ρ, θ (ρ = constant is an ellipse in the $\boldsymbol{\kappa}$-plane) through the transformation

$$\boldsymbol{\kappa} = (\lambda, \mu) = \left(\frac{LM}{PQ}\right)^{\frac{1}{4}} \rho\left(L^{-\frac{1}{2}}\cos\theta, M^{-\frac{1}{2}}\sin\theta\right) \tag{4.6}$$

and similarly for $\boldsymbol{\kappa}'$ and $\boldsymbol{\kappa}''$, so that (3.24) transforms to

$$\sigma = 2\left(\frac{LM}{PQ}\right)^{\frac{1}{2}}(\rho'^2 - \rho''^2). \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.4) and carrying out the integration with respect to ρ'', we obtain

$$G_{pqr} = C^2(LMPQ)^{-\frac{1}{2}} I_{pqr}(\rho, \theta; \gamma), \tag{4.8}$$

where
$$\gamma = \left(\frac{MP}{LQ}\right)^{\frac{1}{2}}, \tag{4.9}$$

$$I_{pqr} = 16\pi \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_0^{\infty} \exp\left[-\tfrac{1}{2}\rho\rho'\phi_{pqr}(\theta, \theta', \theta'') - \tfrac{1}{2}\rho'^2\psi_{pqr}(\theta', \theta'')\right]\rho'\,\mathrm{d}\rho'\,\mathrm{d}\theta'\,\mathrm{d}\theta'', \tag{4.10}$$

$$\phi_{pqr} = \gamma(C_1 cc' + C_2 cc'') + \gamma^{-1}(C_1 ss' + C_2 ss''). \tag{4.11}$$

$$\psi_{pqr} = \gamma(C_{11} c'^2 + C_{12} c'c'' + C_{22} c''^2) + \gamma^{-1}(C_{11} s'^2 + C_{12} s's'' + C_{22} s''^2), \tag{4.12}$$

$c \equiv \cos\theta$, $s \equiv \sin\theta$ and similarly for θ' and θ''.

The integration with respect to ρ' in (4.10), which yields a complementary error function for $\rho > 0$, is elementary in the important special case $\rho = 0$ and yields

$$I_{pqr}(\gamma) = 16\pi \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{d\theta'\, d\theta''}{\psi_{pqr}(\theta', \theta'')} \quad (\boldsymbol{\kappa} = 0), \tag{4.13}$$

which may be reduced to (Appendix B)

$$I_{pqr}(\gamma) = 128\pi^2 (R^2 + S^2\mu^2)^{-\frac{1}{2}} K[S\mu(R^2 + S^2\mu^2)^{-\frac{1}{2}}], \tag{4.14}$$

where
$$R = [(C_{11} + C_{22})^2 - C_{12}^2]^{\frac{1}{2}}, \quad S = (C_{11}C_{22} - \tfrac{1}{4}C_{12}^2)^{\frac{1}{2}}, \tag{4.15a,b}$$

$$\mu \equiv \left|\gamma - \frac{1}{\gamma}\right| = \left|\left(\frac{MP}{LQ}\right)^{\frac{1}{2}} - \left(\frac{LQ}{MP}\right)^{\frac{1}{2}}\right|, \tag{4.16}$$

and $K(k)$ is a complete elliptic integral of modulus k. The function

$$I(\gamma) \equiv I_{134} + I_{234} - I_{123} - I_{124}, \tag{4.17}$$

in terms of which (4.2) reduces to

$$N(0, \tau) = e^{-2\alpha\tau}[\hat{N}(0) + C^2 (LMPQ)^{-\frac{1}{2}} I(\gamma) \hat{N}^2(0)\, T(\tau)], \tag{4.18}$$

is plotted in figure 1. The limiting values are given by

$$I_{pqr}(1) = \frac{64\pi^3}{R}, \quad I(1) = 16\pi^3(\tfrac{3}{2} - \surd 2) = 42.56, \tag{4.19a,b}$$

$$I_{pqr} \sim \frac{128\pi^2}{S\mu} \ln\left(\frac{4S\mu}{R}\right), \quad I \sim 267\mu^{-1}(0.750 - \ln\mu) \quad (\mu \uparrow \infty). \tag{4.20a,b}$$

We remark that $I > 0$, and hence energy flows towards the peak, for $0 \leqslant \mu < 1.86$.

If $L < 0$ (4.6) may be replaced by

$$(\lambda, \mu) = \left(\frac{|L|\, M}{PQ}\right)^{\frac{1}{4}} \rho(\pm |L|^{-\frac{1}{2}} \cosh\theta, M^{-\frac{1}{2}} \sinh\theta), \tag{4.21a}$$

where $\rho =$ constant is a hyperbola, and the $\pm$ sign corresponds to those hyperbolae opening towards $\lambda = \pm\infty$; those hyperbolae opening towards $\mu = \pm\infty$ correspond to

$$(\lambda, \mu) = \left(\frac{|L|\, M}{PQ}\right)^{\frac{1}{2}} \rho(|L|^{-\frac{1}{2}} \sinh\theta, \pm M^{-\frac{1}{2}} \cosh\theta). \tag{4.21b}$$

Proceeding as above then leads to a representation of $I_{pqr}(\gamma)$ that comprises eight integrals, corresponding to the four possible sign combinations in the $(\boldsymbol{\kappa}', \boldsymbol{\kappa}'')$-plane and the complementary transformations (4.21 a, b). The end result, after a reduction similar to that in Appendix B, is (cf. (4.14))

$$I_{pqr}(\gamma) = \begin{cases} 128\pi^2 R^{-1} K([1 - (S\mu_*/R)^2]^{\frac{1}{2}}) \\ 128\pi^2 (S\mu_*)^{-1} K([1 - (R/S\mu_*)^2]^{\frac{1}{2}}) \end{cases}, \quad \frac{S\mu_*}{R} \lessgtr 1, \tag{4.22}$$

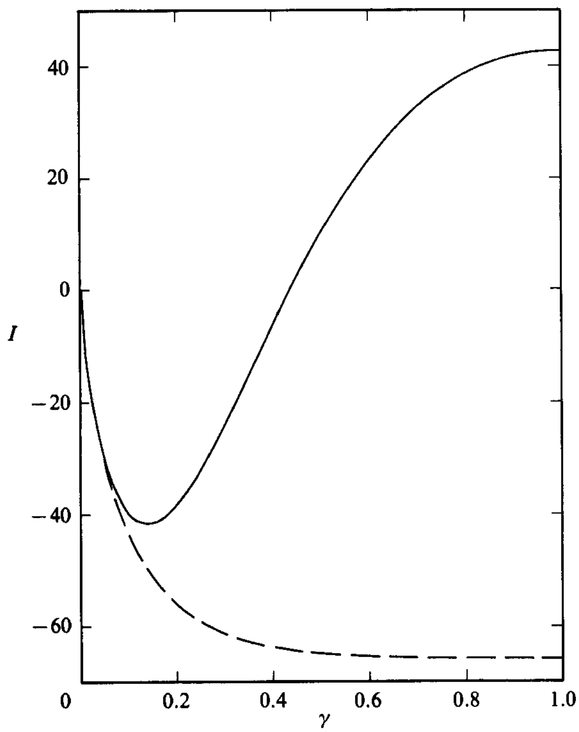

FIGURE 1. The integral $I(\gamma)$, as given by (4.17) and either (4.14) for $L>0$ (——) or (4.22) for $L<0$ (– – –).

where
$$\mu_* \equiv |\gamma| + |\gamma|^{-1} = \left(\frac{MP}{|L|\,Q}\right)^{\frac{1}{2}} + \left(\frac{|L|\,Q}{MP}\right)^{\frac{1}{2}}, \tag{4.23}$$

and R and S are given by (4.15). The limiting values are given by

$$I_{pqr}(1) = \frac{128\pi^2}{R} K\left(\left[1-\left(\frac{2S}{R}\right)^2\right]^{\frac{1}{2}}\right), \tag{4.24}$$

which yields $I(1) = -65.814$, in agreement with Longuet-Higgins's numerical integration (after incorporating a missing factor of 2), and (cf. (4.20))

$$I_{pqr} \sim \frac{128\pi^2}{S\mu_*} \ln\left(\frac{4S\mu_*}{R}\right) \quad (\mu_* \uparrow \infty). \tag{4.25}$$

$I(\gamma)$, as defined by (4.17) (note that $I(\gamma) = -H(|\gamma|^2)$ in Longuet-Higgins's notation) is plotted in figure 1.

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Appendix A. Momentum and energy

The dimensionless momentum and energy of the envelope may be placed in the forms

$$\boldsymbol{M} = \tfrac{1}{4}\mathrm{i}\langle \mathscr{A}\,\nabla\mathscr{A}^* - \mathscr{A}^*\nabla\mathscr{A}\rangle = \tfrac{1}{2}\boldsymbol{\kappa}_n\,\overline{a_n\,a_n^*} \sim \iint \boldsymbol{\kappa}N(\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa} \tag{A 1}$$

and

$$E = \tfrac{1}{4}\mathrm{i}\langle \mathscr{A}^*\mathscr{A}_\tau - \mathscr{A}\mathscr{A}_\tau^*\rangle = \tfrac{1}{2}\sigma_n\,\overline{a_n\,a_n^*} \sim \iint \sigma N(\boldsymbol{\kappa})\,\mathrm{d}\boldsymbol{\kappa}, \tag{A 2}$$

where $\langle\ \rangle$ signifies a spatial average and $\nabla \equiv (\partial_\xi, \partial_\eta)$. Multiplying (3.3) through by $\boldsymbol{\kappa}_n\,a_n^*$, proceeding as in the derivation of (3.10), permuting (j, l, m, n), and invoking $\boldsymbol{\kappa}_j + \boldsymbol{\kappa}_l = \boldsymbol{\kappa}_m + \boldsymbol{\kappa}_n$, we find that the momentum decays according to

$$\boldsymbol{M} = \boldsymbol{M}_0\,\mathrm{e}^{-2\alpha\tau}. \tag{A 3}$$

The evolution equation for the energy is more involved. Differentiating (A 2) with respect to τ, invoking $\dot{\sigma}_n = \dot{\beta}$, and proceeding as in the derivations of (3.10) and (A 3), we obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau} + 2\alpha\right)E = \tfrac{1}{2}\dot{\beta}\overline{a_n\,a_n^*} - C\delta_{jl\overline{mn}}\,\mathrm{Re}\,\{\mathrm{i}\sigma_n\,B_{jl\overline{mn}}\,e_{jl\overline{mn}}\} \tag{A 4a}$$

$$= \dot{\beta}A + \tfrac{1}{4}C\delta_{jl\overline{mn}}\,\mathrm{Re}\,\{\mathrm{i}\sigma_{jl\overline{mn}}\,B_{jl\overline{mn}}\,e_{jl\overline{mn}}\}. \tag{A 4b}$$

To determine β, we multiply (2.10) by $\mathscr{A}^*$, add the complex conjugate of the result, take the spatial average (note that $\langle \mathscr{A}^*\mathscr{A}_{\xi\xi}\rangle = -\langle \mathscr{A}_\xi\mathscr{A}_\xi^*\rangle$ and $\langle \mathscr{A}^*\mathscr{A}_{\eta\eta}\rangle = -\langle \mathscr{A}_\eta\mathscr{A}_\eta^*\rangle$), and invoke the first equality in (A 2) to obtain the alternative representation

$$E = \tfrac{1}{2}\langle L\mathscr{A}_\xi\mathscr{A}_\xi^* + M\mathscr{A}_\eta\mathscr{A}_\eta^* + C\mathscr{A}^2\mathscr{A}^{*2}\rangle \tag{A 5a}$$

$$= \tfrac{1}{2}[(L\lambda_n^2 + M\mu_n^2)\,\overline{a_n\,a_n^*} + C\delta_{jl\overline{mn}}\,B_{jl\overline{mn}}\,e_{jl\overline{mn}}], \tag{A 5b}$$

where $B_{jl\overline{mn}}$ is defined by (3.12). Comparing (A 5*b*) with (A 2) and invoking (3.2), we obtain

$$\beta\overline{a_n\,a_n^*} = C\delta_{jl\overline{mn}}\,B_{jl\overline{mn}}\,e_{jl\overline{mn}}. \tag{A 6}$$

Substituting (3.14) into (A 6) and invoking $\overline{a_n\,a_n^*} = 2A$, we obtain the first approximation

$$\beta = 4CA.\dagger \tag{A 7}$$

Substituting (A 7) into (A 4*b*), remarking that the second term on the right-hand side thereof vanishes (since the quadruple sum is real) in this same approximation, and invoking (3.11), we obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau} + 2\alpha\right)E = -8\alpha C\hat{A}^2\,\mathrm{e}^{-4\alpha\tau}, \tag{A 8}$$

which admits the integral

$$E = (E_0 - 4CA_0^2)\,\mathrm{e}^{-2\alpha\tau} + 4CA_0^2\,\mathrm{e}^{-4\alpha\tau}, \tag{A 9}$$

where $A_0 \equiv \hat{A}$ and E_0 are the initial values of A and E.

We emphasize that (3.11) and (A 3) for the decay of the action and momentum do not, but (A 9) does, depend on the approximation of uncorrelated phases.

† (A 7) does *not* hold for a non-random wave, for which $\beta = 2CA$.

Appendix B. Reduction of I_{pqr}

We require

$$I = 16\pi \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta' \, \mathrm{d}\theta''}{\psi(\theta', \theta'')}, \tag{B 1}$$

where
$$\begin{aligned} 2\psi(\theta', \theta'') = (\gamma+\gamma^{-1})(C_{11}+C_{22}) + (\gamma-\gamma^{-1})(C_{11}\cos 2\theta' + C_{22}\cos 2\theta'') \\ + C_{12}[(\gamma+\gamma^{-1})\cos(\theta'-\theta'') + (\gamma-\gamma^{-1})\cos(\theta'+\theta'')]. \end{aligned} \tag{B 2}$$

Converting the integrals over the second, third and fourth quadrants of the (θ', θ'')-plane to integrals over the first quadrant and introducing

$$\alpha = \theta'+\theta'', \quad \beta = -\theta'+\theta'' \tag{B 3}$$

and
$$2\hat{\psi}(\alpha, \beta) \equiv 2\psi[\tfrac{1}{2}(\alpha-\beta), \tfrac{1}{2}(\alpha+\beta)] \tag{B 4a}$$
$$\begin{aligned} = (\gamma+\gamma^{-1})(C_{11}+C_{22}) + (\gamma-\gamma^{-1})[C_{11}\cos(\alpha-\beta) + C_{22}\cos(\alpha+\beta)] \\ + C_{12}[(\gamma+\gamma^{-1})\cos\beta + (\gamma-\gamma^{-1})\cos\alpha] \end{aligned} \tag{B 4b}$$
$$\equiv a + b\cos\beta + c\sin\beta, \tag{B 4c}$$

we proceed through the sequence

$$I = 32\pi \int_0^{\pi} \int_0^{\pi} [\psi^{-1}(\theta', -\theta'') + \psi^{-1}(\theta', \theta'')] \, \mathrm{d}\theta' \, \mathrm{d}\theta'' \tag{B 5a}$$
$$= 32\pi \int_0^{\pi} \mathrm{d}\alpha \int_0^{\alpha} \mathrm{d}\beta [\hat{\psi}^{-1}(\alpha, \beta) + \hat{\psi}^{-1}(\alpha, -\beta) + \hat{\psi}^{-1}(\beta, \alpha) + \hat{\psi}^{-1}(-\beta, \alpha)] \tag{B 5b}$$
$$= 32\pi \int_0^{\pi} \mathrm{d}\beta \int_{\beta}^{\pi} \mathrm{d}\alpha [\hat{\psi}^{-1}(\alpha, \beta) + \hat{\psi}^{-1}(\alpha, -\beta) + \hat{\psi}^{-1}(\beta, \alpha) + \hat{\psi}^{-1}(-\beta. \alpha)] \tag{B 5c}$$
$$= 32\pi \int_0^{\pi} \mathrm{d}\alpha \int_0^{\pi} \mathrm{d}\beta [\hat{\psi}^{-1}(\alpha, \beta) + \hat{\psi}^{-1}(\alpha, -\beta)] \tag{B 5d}$$
$$= 128\pi^2 \int_0^{\pi} (a^2-b^2-c^2)^{-\frac{1}{2}} \mathrm{d}\alpha \equiv 128\pi^2 \int_0^{\pi} (A + B\sin^2\alpha)^{-\frac{1}{2}} \mathrm{d}\alpha \tag{B 5e}$$
$$= \frac{256\pi^2}{(A+B)^{\frac{1}{2}}} K\left[\left(\frac{B}{A+B}\right)^{\frac{1}{2}}\right], \tag{B 5f}$$

which, after identifying A and B through (B 4), yields (4.14).

Appendix C. Modulational instability

A particular solution of (2.10) that describes an envelope of slowly decaying amplitude, wavenumber $\epsilon k_0(\lambda, \mu)$ and slowly varying frequency $\epsilon^2 \omega_0 \sigma(\tau)$ is given by (cf. (3.1))

$$\mathscr{A} = \mathscr{A}_0 \exp\left\{-\alpha\tau + \mathrm{i}\left[\lambda\xi + \mu\eta - \int_0^{\tau} \sigma(\tau)\, \mathrm{d}\tau\right]\right\} \equiv \mathscr{A}^{(0)}(\xi, \eta, \tau), \tag{C 1}$$

where $\mathscr{A}_0$ is a complex constant and

$$\sigma(\tau) = L\lambda^2 + M\mu^2 + C|\mathscr{A}_0|^2 \, \mathrm{e}^{-2\alpha\tau}. \tag{C 2}$$

We explore the stability of (C 1) by considering the perturbed envelope

$$\mathscr{A} = \mathscr{A}^{(0)}(\xi, \eta, \tau)[1 + a(\xi, \eta, \tau)] \quad (|a| \ll 1). \tag{C 3}$$

Substituting (C 3) into (2.10), invoking (C 1) and neglecting $O(a^2)$, we obtain

$$\{\mathrm{i}(\partial_\tau + 2L\lambda\,\partial_\xi + 2M\mu\,\partial_\eta) + L\,\partial_\xi^2 + M\,\partial_\eta^2\}\,a = 2C\,|\mathscr{A}^{(0)}|^2 a_\mathrm{r}, \tag{C 4}$$

where a_r is the real part of a. Separating the real and imaginary parts of (C 4), posing

$$(a_\mathrm{r}, a_\mathrm{i}) = \mathrm{Re}\,\{(a_1, b_1)\exp[\mathrm{i}(\lambda_1\xi + \mu_1\eta - \int_0^\tau \sigma_1\,\mathrm{d}\tau)]\}, \tag{C 5}$$

where a_1 and b_1 are complex constants, λ_1 and μ_1 are real constants and σ_1 is a possibly complex function of τ, and requiring the determinant of the resulting linear equations for a_1 and b_1 to vanish, we obtain

$$\sigma_1 = 2(L\lambda\lambda_1 + M\mu\mu_1) \pm [(L\lambda_1^2 + M\mu_1^2)^2 + 2C(L\lambda_1^2 + M\mu_1^2)\,|\mathscr{A}^{(0)}|^2]^{\frac{1}{2}}. \tag{C 6}$$

It follows from (C 6) that the perturbation $|\mathscr{A}| - |\mathscr{A}^{(0)}|$ is unstable if and only if

$$|\mathscr{A}^{(0)}|^2 > \frac{\Lambda^2 + \alpha^2}{2|C|\,\Lambda} > 0, \quad \Lambda \equiv -(L\lambda_1^2 + M\mu_1^2)\,\mathrm{sgn}\,C. \tag{C 7a, b}$$

It follows from (2.4) and (2.11) that $\Lambda > 0$ is possible if and only if either

$$0 < k < 0.393 \quad (C > 0, \quad L < 0) \tag{C 8a}$$

or

$$k > 0.707 \quad (C < 0, \quad L > 0), \tag{C 8b}$$

but is not possible if $0.393 < k < 0.707$ $(C > 0, L > 0)$. If either (C 8a) or (C 8b) is satisfied the critical amplitude of $|\mathscr{A}^{(0)}|$ for the modulational instability is determined by $\Lambda = \alpha$ and

$$|\mathscr{A}^{(0)}| > \left|\frac{\alpha}{C}\right|^{\frac{1}{2}} \equiv \mathscr{A}_*. \tag{C 9}$$

The maximum rate of growth of the perturbation is

$$(\sigma_i)_{\max} - \alpha = |C|\,|\mathscr{A}^{(0)}|^2 - \alpha \quad \text{at } \Lambda = |C|\,|\mathscr{A}^{(0)}|^2 > \alpha. \tag{C 10a, b}$$

The maximum value of a perturbation for which $\Lambda = |C|\,|\mathscr{A}_0|^2$ at $\tau = 0$ occurs at

$$\tau_* = -\frac{1}{2\alpha}\ln\left[\frac{1}{2}\left(1 + \left|\frac{\mathscr{A}_*}{\mathscr{A}_0}\right|^4\right)\right] \tag{C 11}$$

and is given by

$$|\mathscr{A} - \mathscr{A}_0|_{\max} = |\mathscr{A}_0|\,(a_1^2 + b_1^2)\,E\left(\left|\frac{\mathscr{A}_0}{\mathscr{A}_*}\right|^2\right), \tag{C 12}$$

where

$$E(x) = \exp\left[x - 1 + x\left(\tan^{-1}\frac{1}{x} - \frac{1}{4}\pi\right)\right] - \tfrac{1}{2}\ln[\tfrac{1}{2}(1 + x^{-2})]. \tag{C 13}$$

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